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The comparison method in asymptotic stability problems☆

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Abstract

The problem of the stability of the unperturbed motion of a non-autonomous system is investigated on the basis of comparison equations. The principle of the quasi-invariance of the positive limit set of a perturbed motion is derived, which enables a new form of the necessary conditions for the stability of an unperturbed motion to be established using Lyapunov vector functions of fixed and constant sign. Problems concerning the stability conditions of unsteady motions and the stabilization of programme motions of mechanical systems are solved.

1. Basic assumptions

Consider the system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{X}(t, \mathbf{0}) \equiv \mathbf{0}, \quad \mathbf{x} \in R^n \tag{1.1}$$

where $\mathbf{X}(t, \mathbf{x}) = (X^1(t, \mathbf{x}), \ldots, X^n\ (t, \mathbf{x})^T)$ is a vector function, defined in the domain

$$\Gamma = R^+ \times G = \{(t, \mathbf{x}) : t \geq 0, \|\mathbf{x}\| < \nu, \nu = \text{const} > 0 \quad \text{or} \quad \nu = +\infty\}$$

and $||\cdot||$ is a certain norm in R^n.

We will assume that the right-hand side of system (1.1) satisfies the Lipsehitz condition with respect to $\mathbf{x}$ uniformly with respect to t, that is, for any compactism $K \subset G$, a number $L = L(K)$ exists such that the inequality

$$\|\mathbf{X}(t, \mathbf{x}_1) - \mathbf{X}(t, \mathbf{x}_2)\| \leq L\|\mathbf{x}_1 - \mathbf{x}_2\| \tag{1.2}$$

is satisfied for any $\mathbf{x}_1, \mathbf{x}_2 \in K$ and any $t \in R^+$.

The family of translations

$$\{\mathbf{X}_\tau(t, \mathbf{x}) = \mathbf{X}(t + \tau, \mathbf{x}), \tau \in R^+\}$$

will then be precompact in a certain compact metric space F. At the same time, it is possible to construct a family of limit systems for system (1.1) in accordance with the following definition.

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Definition 1. (Ref. 1). A function $\mathbf{X}^*(t, \mathbf{x})$, which is specified for a certain sequence $t_j \to +\infty$ by the relation

$$\mathbf{X}^*(t, \mathbf{x}) = \frac{d}{dt}\lim_{j\to\infty}\int_0^t \mathbf{X}_j(\tau, \mathbf{x})d\tau, \quad (t, \mathbf{x}) \in R\times G; \quad \mathbf{X}_j(\tau, \mathbf{x}) = \mathbf{X}(t_j + \tau, \mathbf{x}) \tag{1.3}$$

is called a limit function to $\mathbf{X}(t, \mathbf{x})$. The system of equations

$$\dot{\mathbf{x}} = \mathbf{X}^*(t, \mathbf{x}) \tag{1.4}$$

is called the limit system to the initial system (1.1).

Depending on the sequence $t_j \to +\infty$, a whole family of limit systems (1.4), where $\mathbf{X}^* \in F$, corresponds to system (1.1). At the same time, the positive limit set $\omega^+(t_0, \mathbf{x}_0)$ of the solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ of system (1.1), which is defined by the formula

$$\omega^+(t_0, \mathbf{x}_0) = \{\mathbf{p} \in G : \exists t_k \to +\infty, \mathbf{x}(t_k, t_0, \mathbf{x}_0) \to \mathbf{p}\}$$

is quasi-invariant with respect to the family of systems (1.4) (Refs. 1,2).

We will now introduce the class K_1 of vector functions

$$\mathbf{V} = (V^1, V^2, \ldots, V^k)^T, \quad \mathbf{V} : \Gamma \to R^k$$

where $\mathbf{R}^k$ are k-dimensional spaces with norm $||\cdot||_k$, which are bounded and uniformly continuous in each set $R \times K$ in such a way that, for any compactism $K \subset G$, a number $m = m(K) > 0$ exists and, for any $\varepsilon > 0$, $\delta = \delta(\varepsilon, K) > 0$ exist such that

$$\|\mathbf{V}(t, \mathbf{x})\|_k \le m, \quad \|\mathbf{V}(t_2, \mathbf{x}_2) - \mathbf{V}(t_1, \mathbf{x}_1)\|_k < \varepsilon$$

for all $(t, \mathbf{x}) \in R \times K$, $(t_1, \mathbf{x}_1)$, $(t_2, \mathbf{x}_2) \in R \times K$ which satisfy the inequalities $|t_2 - t_1| < \delta, ||\mathbf{x}_2 - \mathbf{x}_1|| < \delta$.

For each function $\mathbf{V} \in \mathbb{K}_1$, the family of translations

$$\{\mathbf{V}_\tau(t, \mathbf{x}) = \mathbf{V}(t + \tau, \mathbf{x}), \tau \in R^+\}$$

will be precompact in a certain functional metrizable space F_v of continuous functions $\mathbf{V} : \Gamma \to R^k$ with an open-compact topology.[3] Hence, for any sequence $t \to +\infty$, a sequence $t_{l_j} \to +\infty$ and a function $\mathbf{V}^* \in F_v$ exist such that the sequence of translations

$$\{\mathbf{V}_j(t, \mathbf{x}) = \mathbf{V}(t_{l_j} + t, \mathbf{x})\}$$

will converge to $\mathbf{V}^*(t, \mathbf{x})$ in the space F_v and, in fact, the convergence will be uniform with respect to $(t, \mathbf{x}) \in [-\beta, \beta] \times K$ for each number $\beta > 0$ and each compact set $K \subset G$.

We also introduce the analogous classes $\mathbb{K}_2$ and $\mathbb{K}_3$ of vector functions

$$\mathbf{U} : R \times R^k \to R^k \text{ and } \mathbf{W} : R \times G \times R^k \to R^k$$

which are bounded and uniformly continuous with respect to $(t, \mathbf{u}) \in R \times K_2$ and $(t, \mathbf{x}, \mathbf{u}) \in R \times K_1 \times K_2$ for any compact sets $K_1 \subset G$ and $K_2 \subset R^k$. Here, it will additionally be assumed that each function $\mathbf{U} \in \mathbb{K}_2$ is continuously differentiable with respect to $\mathbf{u}$.

Correspondingly, families of limit functions $\{\mathbf{V}^*\}$, $\{\mathbf{U}^*\}$, $\{\mathbf{W}^*\}$ can be constructed for each of the functions $\mathbf{V} \in \mathbb{K}_1$, $\mathbf{U} \in \mathbb{K}_2$, $\mathbf{W} \in \mathbb{K}_3$ and the limit sets $\{(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)\}$ can exist for specific sequences $t_j \to +\infty$.

2. The problem of the localization of a positive limit set $\omega^+(t_0, \mathbf{x}_0)$

We will consider the problem of the limiting behaviour of the solution of system (1.1) by using a continuously differentiable Lyapunov vector function $\mathbf{V} \in \mathbb{K}_1$.[4]

Suppose that, for system (1.1), a function $\mathbf{V}\in\mathbb{K}_1$, $\mathbf{V}\in\mathbf{C}^1$, $\mathbf{V}(t,\mathbf{0})\equiv\mathbf{0}$ exists whose derivative, by virtue of this system, can be represented in the form

$$\dot{\mathbf{V}}(t,\mathbf{x}) = \mathbf{U}(t,\mathbf{V}(t,\mathbf{x}))+\mathbf{W}(t,\mathbf{x},\mathbf{V}(t,\mathbf{x})),\quad \mathbf{U}(t,\mathbf{0})\equiv\mathbf{0},\quad \mathbf{W}(t,\mathbf{0},\mathbf{0})\equiv\mathbf{0} \tag{2.1}$$

where the function $\mathbf{U}=\mathbf{U}(t,\mathbf{u})$, $\mathbf{U}\in\mathbb{K}_2$ is quasi-monotonic and continuously differentiable with respect to $\mathbf{u}\in R^k$, $\partial\mathbf{U}/\partial\mathbf{u}\in\mathbb{K}_2$, the function $\mathbf{W}\in\mathbb{K}_3$, and $\mathbf{W}(t,\mathbf{x},\mathbf{u})\le\mathbf{0}$ for any $(t,\mathbf{x},\mathbf{u})\in R\times G\times R^k$.

It follows from representation (2.1) that $\mathbf{V}(t,\mathbf{x})$ is a comparison vector function and that the system

$$\dot{\mathbf{u}} = \mathbf{U}(t,\mathbf{u}) \tag{2.2}$$

will be a comparison system.[4]

If $\mathbf{V}=\mathbf{V}(t,\mathbf{x})$ is a function which satisfies Eq. (2.1) where $\mathbf{V}(t_0,\mathbf{x}_0)=\mathbf{V}_0$ and $\mathbf{u}=\mathbf{u}(t,t_0,\mathbf{V}_0)$ is a solution of system (2.2) defined in the interval $[t_0,t_0+\beta)$ $\beta>0$, then the inequality

$$\mathbf{V}(t,\mathbf{x}(t,t_0,\mathbf{x}_0))\le\mathbf{u}(t,t_0,\mathbf{V}_0),\quad \forall t\in[t_0,t_0+\beta)$$

is satisfied on the solution $\mathbf{x}=\mathbf{x}(t,t_0,\mathbf{x}_0)$.It follows from the condition $\mathbf{U}\in K_2$ that system (2.2) is precompact[1,2] and a family of limit comparison systems

$$\dot{\mathbf{u}} = \mathbf{U}^*(t,\mathbf{u}),\quad \mathbf{U}^*\in F_u \tag{2.3}$$

can be defined for it.

It follows from the conditions or the right-hand side $\mathbf{U}=\mathbf{U}(t,\mathbf{u})$ of system (2.2) that the solutions of this system $\mathbf{u}=\mathbf{u}(t,t_0,\mathbf{u}_0)$ are continuously differentiable with respect to $(t_0,\mathbf{u}_0)\in R^+\times R^k$. It follows from the property that $\mathbf{u}(t,t_0,\mathbf{u}_0)$ is non-decreasing with respect to $\mathbf{u}_0$[4] that the matrix

$$\mathbf{\Phi}(t,t_0,\mathbf{u}_0) = \frac{\partial\mathbf{u}(t,t_0,\mathbf{u}_0)}{\partial\mathbf{u}_0}$$

is the non-negative normalized $\mathbf{\Phi}(t_0,t_0,\mathbf{u}_0)=\mathbf{I}$ ($\mathbf{I}$ is the identity matrix) fundamental matrix for the linear variational system

$$\dot{\mathbf{y}} = \mathbf{H}(t,t_0,\mathbf{u}_0)\mathbf{y},\quad \mathbf{H} = \left.\frac{\partial\mathbf{U}(t,\mathbf{u})}{\partial\mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}(t,t_0,\mathbf{u}_0)}$$

Henceforth, it will be assumed that the comparison system (2.2) is such that the matrix $\mathbf{\Phi}(t,t_0,\mathbf{u}_0)$ has the following property: for any compactism $K\in R^k$, numbers $M(K)$ and $\alpha(K)>0$ exist such that the inequalities

$$\|\mathbf{\Phi}(t,t_0,\mathbf{u}_0)\|\le M(K),\quad \det\mathbf{\Phi}(t,t_0,\mathbf{u}_0)\ge\alpha(K) \tag{2.4}$$

hold for any $(t,t_0,\mathbf{u}_0)\in R^+\times R^+\times K$.

Suppose $\mathbf{x}=\mathbf{x}(t,t_0,\mathbf{x}_0)$ is some solution of system (1.1) which is bounded by the compactism $K_0\subset G$, $\mathbf{x}(t,t_0,\mathbf{x}_0)\in K_0$ for any $t\ge t_0$ and $\omega(t_0,\mathbf{x}_0)$ is the positive limit set of this solution.

On the basis of Alekseyev's[5] we obtain the following relation between the value

$$\mathbf{V}[t] = \mathbf{V}(t,\mathbf{x}[t]) = \mathbf{V}(t,\mathbf{x}(t,t_0,\mathbf{x}_0))$$

of the function $\mathbf{V}(t,\mathbf{x})$ for the solution $\mathbf{x}=\mathbf{x}[t]=\mathbf{x}(t,t_0,\mathbf{x}_0)$ and the solution

$$\mathbf{u} = \mathbf{u}[t] = \mathbf{u}(t,t_0,\mathbf{V}_0),\quad \mathbf{V}_0 = \mathbf{V}(t_0,\mathbf{x}_0)$$

of the comparison system (2.2)

$$\mathbf{V}(t,\mathbf{x}[t]) = \mathbf{u}[t]+\int_{t_0}^{t}\mathbf{\Phi}(t,\tau,\mathbf{V}(\tau,\mathbf{x}[\tau]))\cdot\mathbf{W}(\tau,\mathbf{x}[\tau],\mathbf{V}(\tau,\mathbf{x}[\tau]))d\tau \tag{2.5}$$

We shall assume that the function $\mathbf{V}(t, \mathbf{x})$ has a lower bound in the set $R^+ \times K_0$ and that the solution of system (2.2) $\mathbf{u}[t]$ has an upper bound for all $t \geq t_0$. Then, constants $\alpha > 0$, $\beta_0 > 0$ exist in accordance with conditions (2.4) such that

$$\beta_0 \geq \sum_{i=1}^{k} (u^i[t] - V^i[t]) \geq -\alpha_0 \sum_{j=1}^{k} \int_{t_0}^{t} W^j(\tau, \mathbf{x}[\tau], \mathbf{V}[\tau]) d\tau \geq 0, \quad \forall t \geq t_0$$

It follows from this that

$$\lim_{t \to +\infty} \mathbf{W}(t, \mathbf{x}[t], \mathbf{V}(t, \mathbf{x}[t])) = \mathbf{0} \tag{2.6}$$

Suppose $\mathbf{p} \in \omega^+(t_0, \mathbf{x}_0)$ is a limit point, defined by the sequence

$$\mathbf{x}(t_j, t_0, \mathbf{x}_0) \to \mathbf{p} \text{ when } t_j \to +\infty$$

We choose the subsequence $t_j \to +\infty$ for which the following convergences hold

$$\mathbf{X}(t_{j_l} + t, \mathbf{x}) \to \mathbf{X}^*(t, \mathbf{x}), \quad \mathbf{U}(t_{j_l} + t, \mathbf{x}) \to \mathbf{U}^*(t, \mathbf{x}), \quad \mathbf{W}(t_{j_l} + t, \mathbf{x}, \mathbf{u}) \to \mathbf{W}^*(t, \mathbf{x}, \mathbf{u})$$

In the same way as taking the limit mentioned earlier,[6] we find that the convergences

$$\mathbf{x}[t_{j_l} + t] \to \mathbf{x}^*[t], \quad \mathbf{u}[t_{j_l} + t] \to \mathbf{u}^*[t]$$

hold uniformly with respect to $t \in [-\beta, \beta]$ for each $\beta > 0$, where $\mathbf{x}*[t] = \mathbf{x}*(t, 0, \mathbf{p})$, $\mathbf{u}*[t] = \mathbf{u}*(t, 0, \mathbf{u}_0^*)$, $\mathbf{u}_0^* = \mathbf{V}*(0, \mathbf{p})$ are corresponding solutions of systems (1.4) and (2.3). At the same time, from relations (2.5) and (2.6), we obtain

$$\mathbf{V}^*(t, \mathbf{x}^*[t]) = \mathbf{u}^*[t], \quad \mathbf{W}^*(t, \mathbf{x}^*[t], \mathbf{V}^*[t]) = \mathbf{0}, \quad \forall t \in R$$

We thereby have the following theorem.

Theorem 1. *We assume that*

1) *a Lyapunov vector function* $\mathbf{V} = \mathbf{V}(t, \mathbf{x})$, $\mathbf{V} \in \mathbb{K}_1$ *exists which satisfies the differential equality* (2.1);
2) *solutions of the comparison system* (2.2) *satisfy condition* (2.4);
3) *the solution* $\mathbf{x}(t, t_0, \mathbf{x}_0)$ *of system* (1.1) *is bounded by a certain compactism* $K \subset \Gamma$ *for all* $t \geq t_0$;
4) *the solution* $\mathbf{u}(t) = \mathbf{u}(t, t_0, \mathbf{V}_0)$ *of the comparison system* (2.2), *where* $\mathbf{V}_0 = \mathbf{V}(t_0, \mathbf{x}_0)$, *is bounded for all* $t \geq t_0$.

Then, a set of limit functions $(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)$ exists for any limit point $\mathbf{p} \in \omega^+(t_0, \mathbf{x}_0)$ such that the solution $\mathbf{x} = \mathbf{x}^*(t, \mathbf{p})$ of system (1.4) with the initial condition $\mathbf{x}^*(0, \mathbf{p}) = \mathbf{p}$ satisfies the relations

$$\mathbf{x}^*(t, \mathbf{p}) \in \omega^+(t_0, \mathbf{x}_0)$$

$$\mathbf{x}^*(t, \mathbf{p}) \in \{\mathbf{V}^*(t, \mathbf{x}) = \mathbf{u}^*(t)\} \cap \{\mathbf{W}^*(t, \mathbf{x}, \mathbf{u}^*(t)) = \mathbf{0}\}, \quad \forall t \in R$$

where $\mathbf{u}^*(t)$ is the solution of the limit comparison system (2.3) with the initial condition $\mathbf{u}^*(0) = \mathbf{V}^*(0, \mathbf{p})$.

The theorem which has been proved is a theorem on the localization of a positive limit set on the basis of a Lyapunov vector function and a comparison system. It is a development of the La Salle invariance principle for an autonomous system[7] and the principle of the quasi-invariance of a non-autonomous system on the basis of a scalar Lyapunov function with a derivative of constant sign.[2]

Example 1. Consider the system of differential equations

$$\begin{aligned}
\dot{x}_1 &= x_1 \exp(-t) + x_2 \sin t - (x_1^3 + x_1 x_2^2) x_3^2 \sin^2 t \\
\dot{x}_2 &= x_1 \sin t + x_2 \exp(-t) - (x_1^2 x_2 + x_2^3) x_3^2 \sin^2 t \\
\dot{x}_3 &= -\frac{\dot{p}(t)}{2p(t)} x_3 + x_4, \quad \dot{x}_4 = -p(t) x_3
\end{aligned} \tag{2.7}$$

where $p(t)$, $0 < p_0 \le p(t) \le p_1$ is a certain continuous function. We take the Lyapunov function in the form

$$V = (V^1, V^2, V^3)^T; \quad V^1 = \frac{1}{2}(x_1 + x_2)^2, \quad V^2 = \frac{1}{2}(x_1 - x_2)^2, \quad V^3 = p(t)x_3^2 + x_4^2$$

Using Theorem 1, it can be shown that each solution $(x_1(t), x_2(t), x_3(t), x_4(t))^T$ of system (2.7) for which $x_3(t) \not\equiv 0$, has a property of the form

$$x_1(t) \to 0, \quad x_2(t) \to 0 \text{ when } t \to +\infty$$

and each other solution, for which $x_3(t) = x_4(t) \equiv 0$, has a property of the form

$$(x_1(t) + x_2(t)) \to \alpha\exp(-\cos t), \ (x_1(t) - x_2(t)) \to \beta\exp(\cos t) \text{ when } t \to +\infty$$

where α and β are certain constants.

3. Theorem on asymptotic stability

We define the scalar function

$$\bar{V}(t, \mathbf{x}) = \sum_{i=1}^{k} V^i(t, \mathbf{x}) \quad \text{or} \quad \bar{V}(t, \mathbf{x}) = \max\{V^1(t, \mathbf{x}), V^2(t, \mathbf{x}), \ldots, V^k(t, \mathbf{x})\}$$

The following result, which develops the theorem on asymptotic stability from (Ref. 4), can be obtained on the basis of Theorem 1.

Theorem 2. *We will assume that a positive-definite Lyapunov vector function* $\mathbf{V} = \mathbf{V}(t, \mathbf{x})$, $\mathbf{V} \in K_1$ *exists such that*

1) *the differential equality* (2.1) *holds*;
2) *the zero solution* $\mathbf{u} = \mathbf{0}$ *of the comparison system* (2.2) *is uniformly stable*;
3) *condition* (2.4) *is satisfied for each bounded solution of the comparison system* (2.2);
4) *for any limit set* $(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)$ *and each bounded solution* $\mathbf{u} = \mathbf{u}^*(t) \ne \mathbf{0}$ *of the limit comparison system* (2.3), *the set*

$$\{\mathbf{V}^*(t, \mathbf{x}) = \mathbf{u}^*(t)\} \cap \{\mathbf{W}^*(t, \mathbf{x}, \mathbf{u}^*(t)) = \mathbf{0}\}$$

does not contain solutions of the limit system (1.4).

Then, the zero solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is uniformly asymptotically stable.

Proof. Conditions *1* and *2* ensure the uniform stability of the zero solution $\mathbf{x} = \mathbf{0}$ of system (1.1). The property of the attraction of the solutions of system (1.1) to the point $\mathbf{x} = \mathbf{0}$ follows from Theorem 1.

We will now prove that the solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is uniformly attracting with respect to $(t_0, \mathbf{x}_0)$, that is, a positive number $\Delta > 0$ exists and a number $T = T(\varepsilon)$ which is independent of t_0 is found for any $\varepsilon > 0$ and $t_0 \ge 0$ which are such that

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon, \quad \forall t \in [t_0 + T, +\infty), \quad \|\mathbf{x}_0\| < \Delta$$

We shall assume that the opposite is true: it is possible to find numbers $\Delta_0 > 0$, $\varepsilon_0 > 0$ such that, for any sequence $T_k \to +\infty$, a sequence $(t_k, \mathbf{x}_k)$, $t_k \ge 0$, $||\mathbf{x}_k|| < \Delta_0$ exists such that

$$\|\mathbf{x}(t_k + T_k, t_k, \mathbf{x}_k)\| \ge \varepsilon_0$$

Without loss of generality, it can be assumed that

$$t_k \to +\infty, \quad \mathbf{x}_k \to \mathbf{x}_0^* \text{ when } k \to +\infty$$

that a sequence $t_k \to +\infty$ defines a limit set $(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)$ since, otherwise, $t_k + T_k/2$ can be taken as t_k and $\mathbf{x}(t_k + T_k/2, t_k, \mathbf{x}_k)$ taken as $\mathbf{x}_k$ and it is possible to pass, if need be, to converging subsequences.

We select the number $\delta_0 = \delta_0(\varepsilon_0) > 0$ from the condition for the uniform stability of the zero solution of system (1.1). Then,

$$\|\mathbf{x}(t + t_k, t_k, \mathbf{x}_k)\| \geq \delta_0 > 0, \quad \forall t \geq 0 \tag{3.1}$$

Taking the limit as $t_k \to +\infty$ in inequality (3.1), we obtain

$$\|\mathbf{x}^*(t, 0, \mathbf{x}_0^*)\| \geq \delta_0 > 0, \quad \forall t \geq 0 \tag{3.2}$$

Suppose $\mathbf{x}_0^{**}$ is the positive limit point of this solution, which is determined by a certain sequence $t_m \to +\infty$. Without loss of generality, we shall assume that the sequence $t_m \to +\infty$ determines the limit set ($\mathbf{X}^{**}$, $\mathbf{V}^{**}$, $\mathbf{U}^{**}$, $\mathbf{W}^{**}$). By virtue of condition *4* of the theorem $\mathbf{x}^{**}(t, 0, \mathbf{x}_0^{**}) \equiv \mathbf{0}$. This contradicts inequality (3.2), which also proves that the solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is uniformly attracting with respect to $(t_0, \mathbf{x}_0)$.

Example 2. We will now consider the problem of the stability of the equilibrium position of a mechanical system with one degree of freedom, the motion of which is described by the equation

$$\ddot{x} + f(t, x)\dot{x} + g(t, x) = 0 \tag{3.3}$$

We shall assume that the continuous functions *f*(*t*; *x*), *g*(*t*, *x*) are such that the conditions

$$\begin{gathered} 0 < f_1 \leq f(t, x) \leq f_2 < +\infty, \quad t \geq 0, \quad x \in R \\ 0 < g_1 x^2 \leq x g(t, x) \leq g_2 x^2, \quad t \geq 0, \quad x \neq 0 \end{gathered} \tag{3.4}$$

are satisfied.

Using the substitution

$$x_1 = x, \quad x_2 = x + 2\dot{x}/f_1$$

Eq. (3.3) can be represented in the form of the system

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{2}f_1 x_1 + \frac{1}{2}f_1 x_2 \\ \dot{x}_2 &= \left[-\frac{1}{2}f_1 + f(t, x_1) - 2\frac{g(t, x_1)}{f_1 x_1}\right]x_1 + \left[\frac{1}{2}f_1 - f(t, x_1)\right]x_2 \end{aligned} \tag{3.5}$$

We take the Lyapunov vector function in the form $V = (|x_1|, |x_2|)^T$ and then obtain the comparison system

$$\begin{aligned} \dot{u}_1 &= -\frac{1}{2}f_1 u_1 + \frac{1}{2}f_1 u_2 \\ \dot{u}_2 &= \left|-\frac{1}{2}f_1 + f(t, x_1(t)) - 2\frac{g(t, x_1(t))}{f_1 x_1(t)}\right|u_1 + \left[\frac{1}{2}f_1 - f(t, x_1(t))\right]u_2 \end{aligned} \tag{3.6}$$

The zero solution $u_1 = u_2 = 0$ of system (3.6) will be uniformly stable if

$$2g_2 \leq f_1^2 \tag{3.7}$$

On applying Theorem 2, we obtain that condition (3.7) is also the condition for the uniform asymptotic stability in the large of the zero equilibrium position of system (3.3). In fact, the set $\{\tilde{V} = \text{const}\}$ does not contain solutions of the limit system

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{2}f_1 x_1 + \frac{1}{2}f_1 x_2 \\ \dot{x}_2 &= \left[-\frac{1}{2}f_1 + f^*(t, x_1) - 2\frac{g^*(t, x_1)}{f_1 x_1}\right]x_1 + \left[\frac{1}{2}f_1 - f^*(t, x_1)\right]x_2 \end{aligned} \tag{3.8}$$

apart from the zero solution $x_1 = x_2 = 0$. Here, $\bar{V} = \max\{|x_1|, |x_2|\}$.

We will illustrate the result which has been obtained using the solution of a problem on the stabilization of the unsteady motion of a physical pendulum[8] as the example.

Suppose the specified motion $\theta = \theta_0(t)$ of the pendulum is created by its variable velocity $\omega(t)$ of rotation about the vertical axis. When the viscous friction forces are taken into account, the equation of motion has the form

$$A\ddot{\theta} = -(mgz_0 + (C-B)\omega^2(t)\cos\theta)\sin\theta - f(t, \theta, \dot{\theta})$$

We introduce the deviation $x = \theta - \theta_0(t)$ of the true motion from the programmed motion and assume that constants f_1 and f_2 exist such that

$$0 < f_1 \le f(t, \theta, \dot{\theta})/\dot{\theta} \le f_2 < +\infty, \quad \forall \dot{\theta} \ne 0$$

The equations of the perturbed motion can be represented in the form of (3.3), where

$$f(t, x)\dot{x} = f(t, \theta_0(t) + x, \dot{\theta}_0(t) + \dot{x}) - f(t, \theta_0(t), \dot{\theta}_0(t))$$
$$g(t, x) = p(t, x)\partial S(x)/\partial x, \quad S(x) = 4(1 - \cos(x/2))$$
$$p(t, x) = A^{-1}(mgz_0\cos(\theta_0(t) + x/2) + (C-B)\omega^2(t)\cos(2\theta_0(t) + x)\cos(x/2))$$

From inequality (3.7), we find the sufficient conditions for uniformly asymptotic stability of the motion $\theta = \theta_0(t)$

$$p(t, 0) = A^{-1}(mgz_0\cos(\theta_0(t)) + (C-B)\omega^2(t)\cos(2\theta_0(t))) \ge p_0 = \text{const} > 0, \ \forall t \ge 0$$

and, moreover

$$2p(t, 0) \le f_1^2/A^2, \quad \forall t \ge 0$$

These are the condition for the second variation of the reduced potential energy of the pendulum to be positive definite for the motion $\theta_0(t)$ and the condition for it to have an upper limit of $f_1^2/(2A^2)$.

Example 3. Let us assume that a rigid body, fixed at the centre of mass O, with principal central axes OX, OY, OZ under the action of a moment

$$M_x = M_x(t), \quad M_y = M_z = 0$$

executes a programmed motion

$$p = p_0(t), \quad q = r = 0 \tag{3.9}$$

around the largest axis of inertia OX ($A < B < C$), where p, q and r are the projections of the angular velocity on to the $OXYZ$ axes.

We now consider the problem of the stabilization of the motion (3.9) by control moments M_1, M_2, M_3 with respect to the OX, OY and OZ axes of the form

$$M_1 = -\alpha x, \quad M_2 = -\alpha_1 y + \alpha_2 z, \quad M_3 = \alpha_3 y - \alpha_4 z$$

where α and $\alpha_i (i = 1, \ldots, 4)$ are certain constants. The equations of the perturbed motion have the form

$$\begin{aligned} A\dot{x} &= (B-C)yz + M_1 \\ B\dot{y} &= (C-A)(x + p_0(t))z + M_2 \\ C\dot{z} &= (A-B)(x + p_0(t))y + M_3 \end{aligned} \tag{3.10}$$

As the Lyapunov function, we take the vector function

$$V = (V_1, V_2)^T; \quad V_1 = |x|, \quad V_2 = (B-A)By^2 + (C-A)Cz^2$$

We find the comparison system

$$\dot{u}_1 = -\alpha u_1 + ku_2, \quad k = \text{const} > 0$$
$$\dot{u}_2 = Mu_2$$

On satisfying the condition

$$M = \max\left\{\frac{1}{B}\left(-2\alpha_1 + \alpha_2 + \alpha_3\frac{C-A}{B-A}\right), \frac{1}{C}\left(-2\alpha_4 + \alpha_3 + \alpha_2\frac{B-A}{C-A}\right)\right\} \le 0$$

the comparison system will be stable. On the basis of Theorem 2, we obtain that the zero solution of system (3.10) is uniformly asymptotically stable as a whole.

4. Lyapunov vector functions of constant sign

We will introduce the following definitions in a similar manner to that described earlier in (Refs. 6,9).

Definition 2. The zero solution $\mathbf{x} = \mathbf{0}$ is stable with respect to the set $\{\bar{V}^*(t, \mathbf{x}) = 0\}$ and the selected limit ensemble $(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)$ if a $\delta = \delta(\varepsilon) > 0$ exists for any $\varepsilon > 0$ such that the condition

$$\mathbf{x} = \mathbf{x}^*(t, \mathbf{x}_0), \quad \mathbf{x}^*(0, \mathbf{x}_0) = \mathbf{x}_0$$
$$\mathbf{x}_0 \in \{\|\mathbf{x}\| < \delta\} \cap \{\bar{V}^*(0, \mathbf{x}) = 0\} \cap \{\mathbf{W}^*(0, \mathbf{x}, \mathbf{0}) = \mathbf{0}\}$$

is satisfied for any solution

$$\|\mathbf{x}^*(t, \mathbf{x}_0)\| < \varepsilon, \quad \forall t \ge 0$$

of system (1.4).

The zero solution $\mathbf{x} = 0$ is asymptotically stable with respect to the set $\{\bar{V} * (t, \mathbf{x}) = 0\}$ and the selected limit ensemble $(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)$ if it is stable and a number $\Delta > 0$ also exists for which, for any $\varepsilon > 0$, $T = T(\varepsilon) > 0$ exists such that, for any solution

$$\mathbf{x} = \mathbf{x}^*(t, \mathbf{x}_0), \quad \mathbf{x}^*(0, \mathbf{x}_0) = \mathbf{x}_0$$
$$\mathbf{x}_0 \in \{\|\mathbf{x}\| < \Delta\} \cap \{\bar{V}^*(0, \mathbf{x}) = 0\} \cap \{\mathbf{W}^*(0, \mathbf{x}, \mathbf{0}) = \mathbf{0}\}$$

of system (1.4), the following condition is satisfied

$$\|\mathbf{x}^*(t, \mathbf{x}_0)\| < \varepsilon, \quad \forall t \ge T$$

Definition 3. The zero solution $\mathbf{x} = 0$ is uniformly stable (uniformly asymptotically stable) with respect to the set $\{\bar{V}^*(t, \mathbf{x}) = 0\}$ and the family of ensembles $\{(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)\}$ if the number $\delta = \delta(\varepsilon) > 0$ in Definition 2 is independent (the numbers $\Delta > 0$ and $T = T(\varepsilon) > 0$ in Definition 2 are independent) of the choice of $(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)$.

Theorem 3. *We assume that a Lyapunov vector function*

$$\mathbf{V} = \mathbf{V}(t, \mathbf{x}) \ge \mathbf{0}, \quad \mathbf{V} \in \mathbb{K}_1$$

exists such that conditions 1–3 of Theorem 2 *are satisfied and, also, the following condition* (*Condition A*): *the zero solution* $\mathbf{x} = \mathbf{0}$ *is uniformly asymptotically stable with respect to the set* $\{\bar{V}^*(t, \mathbf{x}) = 0\}$ *and the family of limiting ensembles* $\{(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)\}$.

The solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is then uniformly stable.

Proof. We will now prove that the solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is stable. We will assume that this is not so. Then, numbers $\varepsilon_0 > 0$ and $t_0 \ge 0$ and sequences $\tau_j \to +\infty$ and $\mathbf{x}_j^0 \to 0$ then exist such that

$$\left\|\mathbf{x}(\tau_j, t_0, \mathbf{x}_j^0)\right\| = \varepsilon_0 \tag{4.1}$$

By virtue of the continuity of the solutions of system (1.1) and condition (4.1), a sequence $t_j \to +\infty$ exists for any number $\varepsilon_1 > 0$ $\varepsilon_1 < \varepsilon_0$ such that the following relations hold

$$\begin{aligned} &\left\|\mathbf{x}(t_j, t_0, \mathbf{x}_j^0)\right\| = \varepsilon_1 \\ &\varepsilon_1 < \left\|\mathbf{x}(t, t_0, \mathbf{x}_j^0)\right\| < \varepsilon_0 \quad \text{for all} \quad t \in (t_j, \tau_j) \end{aligned} \tag{4.2}$$

We now use the notation $\mathbf{x}_j = \mathbf{x}(t_j,\ t_0,\ \mathbf{x}_j^0)$ and consider the solution $\mathbf{x}(t + t_j, t_j, \mathbf{x}_j)$, $t \geq 0$ of system (1.1). Without loss of generality, we shall assume that the sequence $t_j \to +\infty$ is such that $\mathbf{x}_j \to \mathbf{x}_0^*$ when $j \to +\infty$ and that $t_j \to +\infty$ determines the limit ensemble $(\mathbf{X}^*, \mathbf{V}^*, \mathbf{U}^*, \mathbf{W}^*)$. The sequence of solutions $\mathbf{x}(t + t_j, t_j, \mathbf{x}_j)$ of system (1.1) then reduces to the solution $\mathbf{x}^*(t,\ 0,\ \mathbf{x}_0^*)$ of the limit system (1.4) uniformly with respect to $t \in [-\beta; \beta]$ ($\beta > 0$ is an arbitrary number).

From the inequalities

$$0 \leq \bar{V}(t_j, \mathbf{x}_j) \leq \sum_{i=1}^{k} u^i(t_j, t_0, \mathbf{u}_j^0), \quad \mathbf{u}_j^0 = \mathbf{V}(t_0, \mathbf{x}_j^0) \tag{4.3}$$

and the conditions for the stability of the zero solution $\mathbf{u} = \mathbf{0}$ of the comparison system (2.2), we obtain that the following relation holds

$$\bar{V}^*(0, \mathbf{x}_0^*) = 0 \tag{4.4}$$

We will now prove that $\tau_j - t_j \to +\infty$ when $j \to +\infty$. We will assume that this is not so, that is, that a number $\tau = \tau(\varepsilon_1) > 0$ exists such that $\tau_j - t_j \leq \tau(\varepsilon_1)$. Then, on the one hand, a moment $t_1 \in [0, \tau]$ is found such that

$$\left\|\mathbf{x}^*(t_1, 0, \mathbf{x}_0^*)\right\| = \varepsilon_0 \tag{4.5}$$

and, on the other hand, by virtue of Condition A of the theorem, a number $\delta = \delta(\varepsilon) > 0$ exists for $\varepsilon = \varepsilon_0/2 > 0$ such that

$$\left\|\mathbf{x}^*(t, 0, \mathbf{x}_0)\right\| < \varepsilon, \quad \forall t \geq 0$$

We put $\varepsilon_1 = \delta$, and the inequality $||\mathbf{x}^*(t,\ 0,\ \mathbf{x}_0^*)|| < \varepsilon_0/2$ will then be satisfied, which contradicts equality (4.5). The resulting contradiction proves that $\tau_j - t_j \to +\infty$ when $j \to +\infty$. By virtue of Condition A of the theorem, a number $\Delta_1 > 0$ exists and $T = T(\varepsilon)$ exist for any $\varepsilon > 0$ such that the inequality

$$\left\|\mathbf{x}^*(t, 0, \mathbf{x}_0)\right\| < \varepsilon \tag{4.6}$$

holds for all $t \geq T$ and $\mathbf{x}_0$, $||\mathbf{x}_0|| < \Delta_1$. We put $\varepsilon_1 = 3\Delta_1/2$ and $\varepsilon = \varepsilon_1$, then the inequality $||\mathbf{x}^*(t,\ 0,\ \mathbf{x}_0^*)|| \geq \varepsilon_1$ will hold for all $t \geq 0$, which contradicts inequality (4.6). Hence, the stability of the zero solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is proved.

The uniform stability of this solution is proved out using an analogous scheme. We assume that the solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is not uniformly stable. Numbers $\varepsilon_0 > 0$ and $t_0 \geq 0$ and three sequences $\tau_j \to +\infty$, $t_0^j \in R^+$ and $\mathbf{x}_j^0 \to 0$ then exist such that equality (4.1) holds when t_0 is replaced by t_0^j. At the same time, relations (4.2) hold when t_0 is replaced by t_0^j. Using the notation $\mathbf{x}_j = \mathbf{x}(t_j,\ t_0^j,\ \mathbf{x}_j^0)$ and considering the solution $\mathbf{x}(t + t_j, t_j, \mathbf{x}_j)$, $t \geq 0$ of system (1.1), we repeat the preceding arguments. Here, the fact that equality (4.4) holds follows from relations differing from (4.3) by the replacement of t_0 by t_0^j, the conditions for the uniform stability of the zero solution $\mathbf{u} = \mathbf{0}$ of the comparison system (2.2) and from the condition $\mathbf{V} \in \mathbb{K}_1$. The subsequent proof is repeated without any changes.

The following result can be derived as an extension to Theorem 3.

Theorem 4. *We assume that, in addition to the conditions of the preceding theorem, the set*

$$\{\mathbf{V}^*(t,\mathbf{x}) = \mathbf{u}^*(t)\} \cap \{\mathbf{W}^*(t,\mathbf{x},\mathbf{u}^*(t)) = \mathbf{0}\}$$

does not contain solutions of system (1.4) (*here,* $\mathbf{u}^*(t) \neq \mathbf{0}$ *is an arbitrary bounded solution of the limit comparison system* (2.3)).

The solution $\mathbf{x} = \mathbf{0}$ of system (1.1) is then uniformly asymptotically stable.

Theorems 3 and 4 develop well known results[6,9] in the case of a Lyapunov vector function.

Example 4. In Example 3, we assume that, under the action of a moment about the OX axis, a body executes a programmed motion (3.9) around the mean principal central axis of inertia ($B < A < C$). We now consider the problem of the stabilization of this motion by the moments

$$M_1 = -k_1 x, \quad M_2 = -k_2 y, \quad M_3 = -k_3 z$$

where k_1, k_2, k_3 are certain positive constants. The equations of the perturbed motion will have the form of (3.10). The following notation is used

$$\lambda_1 = (C-A)/B, \quad \lambda_2 = (A-B)/C$$

As the Lyapunov function, we take the function

$$v = \frac{1}{2}(\mu_1 y + \mu_2 z)^2 + \frac{1}{2}x^2; \quad \mu_1 = \sqrt{B(A-B)}, \quad \mu_2 = \sqrt{C(C-A)}$$

which is of constant sign.

On calculating the derivative of the Lyapunov function by virtue of the linearized equations of the perturbed motion, we obtain the differential inequality

$$\dot{V} \le \max\{\nu(t), -2k_1^0\}V$$

where

$$\nu(t) = -k_2^0 - k_3^0 + \sqrt{(k_2^0 - k_3^0)^2 + 4\lambda_1\lambda_2 p_0^2(t)}, \quad k_1^0 = \frac{k_1}{A}, \quad k_2^0 = \frac{k_2}{B}, \quad k_3^0 = \frac{k_3}{C}$$

Using Theorem 4, the condition for the stabilization of the programmed motion of the body can be written in the form

$$|p_0(t)| \le \sqrt{\frac{k_2^0 k_3^0}{\lambda_1\lambda_2}}, \quad t \ge 0; \quad |p_0^*(t)| \not\equiv \sqrt{\frac{k_2^0 k_3^0}{\lambda_1\lambda_2}}, \quad t \in R \tag{4.7}$$

where $p_0^*(t)$ is a function which is the limit function to $p_0(t)$.

Note that, in the case of a steady rotation ($p_0 = \text{const}$), condition (4.7) is identical to the condition obtained using the Routh-Hurwitz criterion.

It can be shown that the asymptotic stability of the unsteady rotation in Examples 3 and 4 will be exponential.

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